# High accuracy gravitational red-shift evaluation at INRIM

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***Abstract*—The gravitational redshift is presently one of the main biases in the comparison of primary frequency standard, and it could be a challenge for the forthcoming generation of atomic optical clocks. In this paper we report the evaluation of the gravitational redshift with respect to the Geoid reference at INRIM laboratories, with an accuracy of $1\times10^{-17}$ in terms of relative frequency. The evaluation is based on GPS/levelling measurements and on the use of a local model for the Earth Geoid and its accuracy is limited by the reference potential of the Geoid.**

## I. Introduction

The second in the International System (SI), is realized with an accuracy ranging from 4 to 10 parts in $10^{16}$ by Caesium atomic fountain frequency standards [1-6]. The International Atomic Time (TAI) is defined as a time coordinate in the general relativity framework; its unit is given through the SI second, realized on the rotating Geoid [7]. The position of the frequency standard on the Geoid generates the well known gravitational shift, actually due to the combined effect of gravitational and centripetal potential of the rotating Earth. Even if we could state that the gravitational shift is not presently the main uncertainty contribution for a frequency comparison between Primary Frequency Standards (PFS) at different locations, a different scenario appears when we consider the forthcoming generation of atomic frequency standard, based on optical resonances of either ions or neutral atoms, as their accuracy is expected to reach $1\times10^{-17}$ or lower in the near future. Today it is not possible to have a better accuracy in the realization of the SI second on the rotating Geoid due to theoretical and experimental limits of global geoid models; in the remote comparison of PFS an higher accuracy is achievable when a common local geodetic model can be used for both the PFS. Since 2003 INRIM is regularly participating to the calibration of the TAI with its Cs fountain IEN-CsF1, whose accuracy is at the moment $8\times10^{-16}$ , and it has started the development of a second generation Cs fountain and a neutral Yb optical clock based on the $^1S_0$-$^3P_0$ forbidden transition, expecting an accuracy of parts in $10^{-16}$ for the first and parts in $10^{-17}$ for the second. In collaboration with the Geodesy division of the Politecnico di Torino, we have re-evaluated the orthometric height at INRIM fountain site with an uncertainty of 5 cm yielding an uncertainty in the gravitational red-shift correction of $1\times10^{-17}$, a tenfold improvement with respect to the previously available evaluation.

## II. The Gravitational Redshift: Definitions and Considerations

When a primary frequency standard is used to calibrate TAI, its frequency needs to be corrected for the gravitational shift with respect to the Geoid. The Geoid is the gravity field equipotential surface which best approximates the mean sea surface; as the mean sea surface is very dynamical because of several effects (moon tides for example), the Geoid can differ from the mean sea surface at different locations and times.

The gravitational red-shift in a frequency standard in the low field limit ( $W/c^2 << 1$ ) is given by the formula [8]:

$$\frac{f_0 - f(\bar{r})}{f_0} = \frac{W(\bar{r}) - W_0}{c^2} \tag{1}$$

where $f_0$ is the atomic frequency, unperturbed and on the Geoid, $r$ is the position vector of the standard in a given reference, $f(r)$ is the clock frequency shifted by gravity, $W(r)$ is the value of the gravity potential, $W_0$ is the reference potential at the Geoid surface and $c$ is the speed of light.

The gravity field potential $W(r)$ has two components, the gravitational $V(r)$ and the centripetal $\Phi(r)$:

$$W(\bar{r}) = V(\bar{r}) + \Phi(\bar{r}) \tag{2}$$

Using a spherical function expansion [9] we obtain:

$$W(\bar{r}) = W(r, \vartheta, \lambda) = \tag{3}$$

$$= \frac{GM}{r}\left(1 + \sum_{n=0}^{\infty}\left(\frac{a}{r}\right)^n \sum_{m=0}^{n}(C_{nm}\cos m\lambda + S_{nm}\sin m\lambda)P_{nm}(\sin\vartheta)\right) + \frac{1}{2}\omega_0^2 r^2 \cos^2\vartheta$$

where $r$, $\theta$, $\lambda$ are respectively the radius, the latitude and the longitude coordinate; $G$ is the universal gravitational constant ($G=6.673(10)\times 10^{11}$), $M$ is the Earth mass, $a$ is the equatorial radius (regarded as constant), $\omega_0$ is the actual Earth angular frequency, $C_{nm}$, $S_{nm}$ are the fully normalized potential coefficients and $P_{nm}$ are the fully-normalized Legendre associated function.

The accuracy of the model depends both on the number of evaluated coefficients $C_{nm}$, $S_{nm}$, and on the accuracy of each coefficient. The coefficient evaluation uses data coming from different measurements, such as satellite motion tracking (data ranging), geodetic measurements, gravimetry, GPS time measurements, levelling campaigns.

Considering the actual gravity potential, $W(r)$, we introduce the following definition

$$W(\bar{r}) = U(\bar{r}) + T(\bar{r}) \tag{4}$$

where $U(r)$ is an equipotential surface called normal gravity field, identified by the Earth ellipsoid and $T(r)$ is called the *anomalous potential*, or *disturbing potential*.

The Earth ellipsoid approximates the Earth shape and defines the normal gravity, by means of two geometrical (the two semi-axis $a$ and $b$ or derived quantities) and two dynamical parameters (the total mass $M$ and the angular frequency $\omega_0$). A very common choice for the geometrical parameters is the pair ($a$, $f$), where $f$ is the flattening of the ellipsoid defined as $f=(a-b)/a$.

Introducing the linear eccentricity $E = \sqrt{a^2 - b^2}$ , and the first eccentricity of the ellipse $e = \sqrt{a^2 - b^2}/b$ , the value $U_0$ of the normal gravity potential on the ellipsoid surface is given by the relation:

$$U_0 = \frac{GM}{E} arctg(e) + \frac{1}{3}\omega_0^2 a^2 \tag{5}$$

where the second term can be regarded as a centripetal effect.

The most important ellipsoid is the World Geodetic Surface 1984 (WGS1984), which is the reference for the GPS satellite navigation system, and it is defined by the following parameters [10,11]:

$$\begin{cases} a = 6378136.6\ (1)\, m \\ f = 1/298.25765\ (1) \\ GM = 3.986004418(8) \times 10^{14} m^3 s^{-2} \\ \omega_0 = 7.292115(.1) \times 10^{-5} rad\ s^{-1} \end{cases} \tag{6}$$

the use of the product $GM$ allows lower uncertainty than the evaluation of the two separated factors $G$ and $M$ and this is the reason of the use of $GM$ as a defining parameter. From the set (6) the prescribed normal gravity potential is

$$U_0^{WGS84} = 62636851(1)\ m^2 s^{-2} \tag{7}$$

Actually, more accurate determinations of $U_0$ exist, leading to a value of [10]

$$U_0^{REF} = 62636855.8(5)\ m^2 s^{-2} \tag{8}$$

It is useful to introduce here the normal gravity on the ellipsoid surface $\gamma_0(\theta)$ defined as

$$\gamma_0 = \gamma_a \frac{1 + \frac{b\gamma_b - a\gamma_a}{a\gamma_a}\sin^2\vartheta}{\sqrt{1 - e^2 \sin^2\vartheta}} \tag{9}$$

where $\gamma_a$ and $\gamma_b$ are the theoretical gravity at the equator and at the poles

$$\begin{aligned} \gamma_a &= \frac{GM}{ab}\left[1 - \frac{3}{2}m - \frac{3}{14}e^2 m\right] \\ \gamma_b &= \frac{GM}{ab}\left[1 + m + \frac{3}{7}e^2 m\right] \\ m &= \frac{\omega^2 a^2 b}{GM} \end{aligned} \tag{10}$$

The definition of a Geoid uses the convention to constrain the value $W_0$ of the actual gravity potential on the Geoid surface to be equal to the normal potential $U_0$ of a reference ellipsoid. From (9), the uncertainty on the $W_0$ value of the Geoid associated to the WGS84 ellipsoid corresponds to a relative frequency uncertainty in the primary frequency standards of $1\times10^{-17}$ in the realization of the SI second on the reference geoid, or $5\times10^{-18}$ if the re-evaluation (8) is considered.

Near the Earth surface, a good approximation for (1) is usually considered, given by the relation [12]:

$$\frac{f_0 - f(\bar{r})}{f_0} \approx \frac{\gamma_0}{c^2} H \tag{11}$$

where $H$ is the height of the frequency standard above the Geoid surface, called *orthometric height*, as discussed in the following paragraph, while $\gamma_0$ value of the normal gravity acceleration on the ellipsoid, definied in (7). Considering that $\gamma_0\ /c^2 \approx 1.09\times10^{-16}$, the uncertainty of 1 m on the orthometric height produces a $1\times10^{-16}$ uncertainty for the standard relative frequency. This is the typical level of accuracy for the gravitational correction in atomic fountains: up to now only the National Institute of Standard and Technology has provided a gravity redshift evaluation with $3\times10^{-17}$ uncertainty by the determination of the PFS position on the Geoid at a 30 cm level [13]. Aiming the goal of $1\times10^{-17}$ accuracy, the determination of the standard height at few decimetres level or better is strictly required, and this is possible within the present status of art in geodesy. The approximation given in (11) is considered acceptable at this level of accuracy.

The Geoid is often expressed in terms of the distance from the reference ellipsoid. A given point on the Earth surface is located by a *geodetic height h*, relative to the ellipsoid, an *orthometric height H* , relative to the Geoid, and a *Geoid undulation N* defined as the difference:

(12) $$N = h - H$$

As pointed out before, the present value of $U_0$ reported in (8), $U_0^{REF}$, is evaluated with higher accuracy than $U_0^{WGS84}$ . This improved value implies that the $W_0$ potential is not equal to the $U_0$ defined on the reference ellipsoid WGS84, this is usually taken into account with the use of a "zero order" undulation $N_0$.

Considering the temporal stability of $W_0$, we may point out that the parameters *GM, E , a* and *e* can be considered as constants while $\omega_0$ needs some more care.

The observations in the period 1978-1999 reported by the IERS (International Earth Rotation and Reference Systems Service), suggest for $\omega_0$ a long term drift:

(13) $$d\omega_0 / dt = (-4.5 \pm 0.1) \times 10^{-22} \ rad \ s^{-2}$$

that corresponds from equation (2) to potential changes of

(14) $$dW_0/dt < 10^{-12} \ m^2 \ s^{-3}$$

and then negligible for the SI second realization at the $10^{-20}$ over 30 years (within the hypothesis of a constant variation of the angular frequency). In the medium-short term, the inter-annual variations of $\omega_0$ can be evaluated on the basis of the sidereal day. The latter shows variations of about hundreds of milliseconds, corresponding to angular frequency fluctuations of

(15) $$d\omega_0 / dt \sim 8 \times 10^{-11} \ rad \ s^{-1} day^{-1}$$

and then

(16) $$\Delta W_0 \sim 0.2 \ m^2 \ s^{-2} \qquad \Delta f/f_0 \sim 2 \times 10^{-18}$$

these fluctuations maybe will be relevant to the realization of an optical clock.

Satellite missions aiming to the determination of the Geoid at centimetric and millimetric level should give in the future also a more detailed evaluation of the temporal evolution of the Earth gravity field.

## III. Orthometric Height at INRIM

The orthometric height at the INRIM location was evaluated in collaboration with the Geodesy Department of the Politecnico di Torino, using two Geoid models, the global model EGM96 [14] and the regional model ITALGEO99 [15], with the aid of geometrical levelling techniques (Military Geographical Institute orthometric height markers) and GPS levelling technique (International Global Navigation Satellite Systems Service sites). ITALGEO99 is a local quasi-Geoid model based on gravimetric measurements developed at the Politecnico di Milano (Italy); and it estimates the values of mean 0.033°×0.033° undulations on a 0.033°×0.033° area, that means, at INRIM latitude, 3,7 km×2,6 km (latitude×longitude) solutions. The uncertainty of the undulation estimations are evaluated to be less than 20 cm [15]. A re-evaluation of ITALGEO99 in 2006 now gives uncertainties smaller than 5 cm on absolute undulation values and smaller than 3 cm on undulation difference between two close locations.

Since 1995 the Italian Military Geographical Institute (IGM) has delivered IGM95 [16], a GPS geodetic reference system composed of 1236 markers distributed all over Italy, obtained with GPS levelling techniques. Since 2006 a re-evaluation of the markers data is available. IGM95 is the Italian WGS84 ellipsoid materialization and provides also the orthometric heights at fixed locations with an estimated uncertainty of 3 cm.

In the neighbourhood of the Politecnico di Torino buildings there are four geodetic markers, useful for our goal; their heights, as evaluated in 2006, are reported in Table 1.

The Time and Frequency Department at INRIM and the Geodesy Department in Politecnico are both official sites of the International Global Navigation Satellite systems Service (IGS) [17]). These products include also the station coordinates evaluation in the WGS84 reference at the millimiter level, allowing an accurate definition of ellipsoid height at Politecnico and at INRIM.

The coordinates of the Politecnico and INRIM IGS stations, named respectively TORI and IENG in the IGS Network are reported in Table 2 (iono-free IGS solution)[17] .

The uncertainty associated to the height difference between the two station is 1 cm.

| Marker | Height (m) |
|---|---|
| IGM95 n.56906 | 246.07 |
| IGM95 n.56906 associated | 248.14 |
| Horizontal Marker Turin n.337 | 245.47 |
| Vertical Marker Turin n.337 | 248.44 |

Table 1. Quoted geodetic markers and respective orthometric height closest to Politecnico of Turin. Uncertainty is less than 3 cm.

| | Latitude | Longitude | Geodetic Height (m) |
|---|---|---|---|
| IENG | 45°00'54.467" N | 7°38'21.842" E | 316.600 |
| TORI | 45°03'48.114" N | 7°39'40.597" E | 310.726 |

Table 2. Coordinates of the two IGS stations IENG (at INRIM) and TORI (at Politecnico)

We provide three different evaluations of the orthometric height at INRIM:

1. Using the geodetic coordinates with respect to WGS84 provided by the IGS and the geoid undulation calculated with the EGM96 global model the following values are obtained:

(17) $N = 48 \pm 1\ m \qquad H_{IENG}=(268.6 \pm 1)\ m$

2.Using the undulation calculated with the regional model ITALGEO99 we obtain the following values

(18) $N = 49,10 \pm 0.05\ m \qquad H_{IENG}=(267.50 \pm 0.05)\ m$

3. The most accurate evaluation of the orthometric height is obtained combining the data coming from IGM95 markers, ITALGEO99 orthometric heights difference products and some direct geometrical distances measurements.

A scheme of the markers used or measured int this evaluation is reported in Fig. 1, where it is detailed the reciprocal positions of the marker and the kind of markers (IGM,GPS, IGS stations or measured markers).

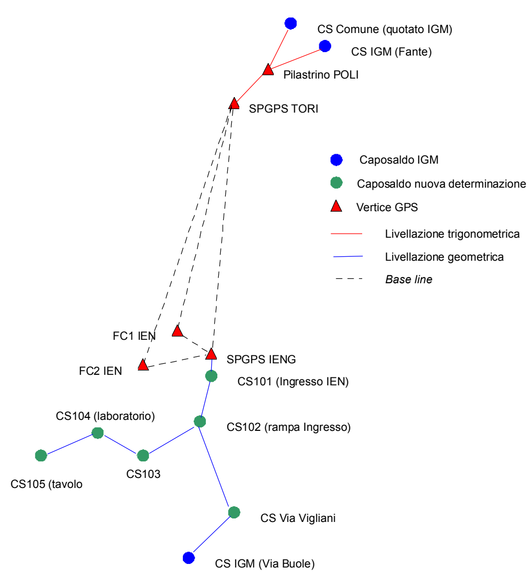


Figure 1. Scheme of the markers used for the local orthometric height at INRIM

In fact, the orthometric height at INRIM IGS station $H_{IENG}$ is equal to

(19) $H_{IENG} = H_{TORI} + \Delta H_{TORI\text{-}IENG}$

Where $H_{TORI}$ is the orthometric height of the Politecnico IGS station and $\Delta H_{TORI\text{-}IENG}$ is the difference of the two heights.

In this case, $H_{TORI}$ is evaluated by a geometrical levelling of the TORI IGS station marker with respect to the IGM markers. The geometrical levelling is performed with a NIKON DTM730 total station, and the uncertainty of this measurement is evaluated to be 3 mm.

$H_{TORI}$ is then, with respect to the markers IGM95 and Turin n.337

$H_{TORI}$ (IGM95) = 261.7022 m

$H_{TORI}$ (Turin n. 337) = 261.6971 m

As the difference between the two evaluation is less then 5 mm, we take the average:

(20) $H_{TORI} = 261{,}70 \pm 0.03$ m

where the uncertainty is dominated by the contribution of the Geodetic Markers.

The difference $\Delta H_{TORI\text{-}IENG}$ between the orthometric height at the two locations is evaluated using the ellipsoid height and the ITALGEO99 model which provides the undulation difference between close locations at 2 cm uncertainty level, that is more accurate than the undulation values provided by the global model.

The ellipsoid heights difference has a 1 cm uncertainty as reported for example in [22].

The difference $\Delta H_{TORI\text{-}IEN}$ is then

$\Delta H_{TORI\text{-}IENG} = H_{TORI} - H_{IENG} = (h_{IENG} - N_{IENG}) - (h_{TORI} - N_{TORI}) = (h_{IENG} - h_{TORI}) - (N_{IENG} - N_{TORI}) = (5.82 \pm 0.02)$ m

and

(21) $H_{IENG} = H_{TORI} + \Delta H_{TORI\text{-}IENG} = 267.52 \pm 0.04$ m

referred to the IENG station marker.

In conclusion, IENG station orthometric height could be evaluated from the weighted average of the three values, and we obtain

(22) $H_{IENG} = 267.51 \pm 0.03$ m

The IENCsF1 laboratory location has been measured with respect to IENG station by geometrical levelling and then we obtain the elevation of the Cs fountain as well:

(23) $H_{IENCsF1} = 239.43 \pm 0.03$ m

The contribution to the gravitational red-shift uncertainty from the orthometric height evaluation in the framework of the ITALGEO99 regional model is less than $3x10^{-18}$. At this level, the main uncertainty contribution is that of the reference potential of the Geoid itself, then $1x10^{-17}$ as reported in (7).

The red-shift correction with respect tto the Geoid to be applied to the SI realized with IEN-CsF1 is then

(24) $\Delta f/f_0 = (-2.610 \pm 0.001\ )\ x\ 10^{-14}$

Considerations should be made about the opportunity of defining an absolute value for the Geoid potential, together with further consideration of its temporal stability, in order to reduce the role of $W_0$ uncertainty in PFS accuracy and comparisons.

## IV. CONCLUSIONS AND PERSPECTIVES

In conclusion, we have evaluated the gravitational red-shift at INRIM within an $1x10^{-17}$ accuracy, limited by the Geoid reference uncertainty. The gravitational red-shift and the Geoid model will set a limit for the comparison among different optical clocks as they will achieve the expected accuracies at the $1x10^{-17}$ level or better.

Presently, other models, i..e. other sets of coefficients for the Geoid field expansion, have been evaluated from satellite missions [18]. These models can improve EGM96 coefficients accuracy, as the uncertainty of the gravity field evaluation is reported to be $2\text{-}3x10^{-17}$ in terms of relative frequency and even further improvements are expected.

Local direct measurements as those used in these paper allow to reach the highest accuracy in the gravity correction evaluation, but it seems that in the next future the improved satellite measurements should offer the best solution to compare directly remote clocks, and even to give useful information about the temporal stability of the local gravity field in the medium term.

In fact, a local model offers the best accuracy for the value of the gravity field, but the comparison or the interpolation among different local models, maybe not overlapping or without common borders introduces an uncertainty higher than that offered by a global model.

The gravity field temporal stability should be considered at this level of accuracy and it will be the subject of further studies.

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